Frobenius Endomorphisms for Koszul Complexes

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1 Rings of Prime Characteristic

2 Homological Tools

3 Upgrading to Koszul Complexes

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Such rings are equipped with a Frobenius endomorphism

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- $\sqrt[p]{x} + \sqrt[p]{y} = \sqrt[p]{x+y}$ and $\sqrt[p]{x} \cdot \sqrt[p]{y} = \sqrt[p]{xy}$, and
- $r \cdot \sqrt[p]{x} = \sqrt[p]{r^p x}$ for all $r \in R$.

Example

Let $k = \mathbb{Z}/p\mathbb{Z}$ and let $R = k[x]$. Then F_*R is generated by $a = \sqrt[p]{x}$ satisfying $a^p = x$ showing

$$F_*R \cong \frac{R[a]}{(a^p - x)} = \bigoplus_{i=0}^{p-1} Ra^i$$

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For $R = k[x]/(x^2)$ $(\sqrt[p]{x})^2 = \sqrt[p]{x^2} = 0$ showing $a = \sqrt[p]{x}$ satisfies $a^2 = 0$ and $a^p = x$. Hence

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We say R is *F-finite* if F_*R is f.g. as an R-module.

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Question

Why does F_*R encode the regularity of R?

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Definition

An R-complex M is a family of R-linear maps

$$\cdots \to M_{n+1} \xrightarrow{\partial_{n+1}^M} M_n \xrightarrow{\partial_n^M} M_{n-1} \to \dots$$

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A chain map $f\colon M \to N$ is a family of R-linear maps

$$\begin{tikzcd}
\dots \arrow[r] & M_{n+1} \arrow[r, "\partial^M_{n+1}"] \arrow[d, "f_{n+1}"] & M_n \arrow[r, "\partial^M_n"] \arrow[d, "f_n"] & M_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & N_{n+1} \arrow[r, "\partial^N_{n+1}"'] & N_n \arrow[r, "\partial^N_n"'] & N_{n-1} \arrow[r] & \dots
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\dots \arrow[r] & N_{n+1} \arrow[r, "\partial_{n+1}^N"'] & N_n \arrow[r, "\partial_n^N"'] & N_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

Example

Every R-module is a complex concentrated in degree $n = 0$ with trivial differential. A chain map of modules is exactly a linear map.

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A projective resolution P of a module M is equivalent to a quasi-isomorphism $P \xrightarrow{\simeq} M$ from a complex P of projectives to M

$$\begin{tikzcd}
\dots \arrow[r] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, two heads] & 0 \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

Definition

The level of M against N, denoted $\mathrm{level}_R^N(M)$, is the smallest integer d such that there is a resolution $X \xrightarrow{\simeq} M$ of the form

$$0 \to X_{d-1} \to X_{d-2} \to \cdots \to X_1 \to X_0 \to 0$$

with each X_i a direct summand of some $N^{\oplus q_i}$. We write $N \Rightarrow M$ if $\mathrm{level}_R^N(M) < \infty$.

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Example

Let R be a Noetherian ring and M a f.g. R-module.

$$\mathrm{level}_R^R(M) < \infty \Leftrightarrow \begin{array}{c} M \text{ has a finite length} \\ \text{projective resolution} \end{array} \Leftrightarrow \mathrm{pd}_R(M) < \infty$$

Heuristic

Let Hdim be a homological dimension $(\mathrm{pd}_R, \text{CI-}\mathrm{dim}_R, \mathrm{Gpd}_R,$ etc.), and let M, N be two modules with $N \Rightarrow M$.
If $\mathrm{Hdim}(N) < \infty$ then $\mathrm{Hdim}(M) < \infty$.

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If $\mathrm{Hdim}(N) < \infty$ then $\mathrm{Hdim}(M) < \infty$.

If G is a generator and $\mathrm{Hdim}(G) < \infty$ then $\mathrm{Hdim}(M) < \infty$ for all f.g. modules, showing G detects homological properties of R itself.

Theorem (Ballard-Iyengar-Lank-Mukhophadyay-Pollitz '25)

Let R be an F-finite ring, then $F_*^e R$ is a generator for $e \gg 0$. Moreover if R is a locally complete intersection then the first pushforward $F_* R$ is a generator.

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1. A local-to-global principal from Benson-Iyengar-Krause '15:

$$F_*^e R \text{ is a generator} \iff \begin{array}{c} F_*^e R_{\mathfrak{p}} \Rightarrow \kappa(\mathfrak{p}) \\ \text{for all } \mathfrak{p} \in \operatorname{Spec}(R) \end{array}$$

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2a. In the general case the level is bounded above by $\operatorname{edim}(R_{\mathfrak{p}}) + 1$ using simplicial algebras.

2b. In the locally complete intersection case cohomological supports and a theorem of Stevenson '14 show the level is finite without constructing an upper bound.

Question

Is $\operatorname{level}_R^{F_*R}(k) \leq \operatorname{edim}(R) + 1$ for an F-finite local complete intersection?

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Let $f \in R$, then the Koszul complex on f over R is the complex

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Quasi-isomorphisms preserve homological data, so we can work with $K^R(f)$ instead of S.

Definition (- '26)

The Frobenius endomorphism for $E = K^R(f)$ is the chain map $F^E \colon E \to E$ given by

$$\begin{tikzcd} Re \arrow[r, "f"] \arrow[d, "f^{p-1} \cdot F^R"'] & R1 \arrow[d, "F^R"] \\ Re \arrow[r, "f"'] & R1 \end{tikzcd}$$

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Letting $A = F_*R$, the Frobenius pushforward of E is the complex $F_*E = K^A(\sqrt[p]{f}\,)$

$$F_*E\colon A\sqrt[p]{e} \xrightarrow{\sqrt[p]{f}} A1$$

The E-algebra structure is given by $e \cdot (-) = \sqrt[p]{f^{p-1}} \cdot \sqrt[p]{e} \cdot (-)$.

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Letting $E = K^R(f)$ and $A = F_*R$ we see

$$F_*E\colon\, A\sqrt[3]{e} \xrightarrow{a^2} A1$$

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Representing the differential and multiplication as a matrix

$$\partial = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & x \\ 1 & 0 & 0 \end{bmatrix} \qquad e \cdot (-) = \begin{bmatrix} 0 & 0 & x^2 \\ x & 0 & 0 \\ 0 & x & 0 \end{bmatrix}$$

Theorem (- '26)

Let $(R, \mathfrak{m}, k)$ be an F-finite regular local ring, $f \in R$, and $E = K^R(f)$. Then $\mathrm{level}_E^{F_*E}(k) \leq \mathrm{edim}(R) + 1$ and F_*E is a generator.

Corollary (-, '26)

Let R be an F-finite ring which is locally hypersurface, then

$$\mathrm{level}_{R_\mathfrak{p}}^{F_*R_\mathfrak{p}}(\kappa(\mathfrak{p})) \leq \mathrm{edim}(R) + 1 < \infty \text{ for all } \mathfrak{p} \in \mathrm{Spec}(R)$$

Thank you!